An Examination of Pre-service Secondary Mathematics Teachers' Conceptions of Angles

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Abstract: The concept of angles is one of the foundational concepts to develop of geometric knowledge, but it remains a difficult concept for students and teachers to grasp. Exiting studies claimed that students' difficulties in learning of the concept of angles are based on learning of the multiple definitions of an angle, describing angles measuring the size of angles, and conceiving different types of angles such as 0-line angles, 1- line angles, and 2-line angles. This study was designed to gain better insight into pre-service secondary mathematics teachers' (PSMTs) mental constructions of the concept of angles from the perspective of **A**ction-**P**rocess-**O**bject-**S**chema (APOS) learning theory. The study also explains what kind of mental constructions of angles is needed in the right triangle context. The four PSMTs were chosen from two courses at a large public university in the Midwest United States. Using Clements' (2000) clinical interview methodology, this study utilized three explanatory interviews to gather evidence of PSMTs' mental constructions of angles and angle measurement. All of the interview data was analyzed using the APOS framework. Consistent with the existing studies, it was found that all PSMTs had a schema for 2-line angles and angle measurement. PSMTs were also less flexible on constructions of 1-line and 0-line angles and angle measurement as it applied to these angles. Additionally, it was also found that although PSMTs do not have a full schema regarding 0-line and 1-line angles and angle measurement, their mental constructions of 1-line and 0-line angles and angle measurement were not required in right triangles, and the schema level for 2-line angles was sufficient for constructions of right triangle context.

Key Words: The Concept of Angles, APOS Learning Theory, Angles and Angle Measurement, Right Triangle, Pre-service Secondary Mathematics Teachers

The concept of angles is a key factor within geometry, and learning the definition of an angle and relationships between an angle and its components is an important step to success in the discipline. Numerous researchers have pointed out that angles, angle measurements, and

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angle rotation concepts are central to the development of geometric knowledge (Browning, Garza-Kling, & Sundling, 2008; Clements & Battista, 1989, 1990; Keiser, 2000, 2004; Mitchelmore & White, 1998, 2000; Moore, 2013, 2014). In addition, the National Council of Teachers of Mathematics (NCTM) Standards (1991, 2000) have stressed the importance of the concept of angles in mathematics curriculum, but it remains a difficult concept for students and teachers to grasp (Clements & Battista, 1989, 1990; Keiser, 2004; Mitchelmore & White, 1998). Students have a variety of difficulties in learning the concept of angles. Researchers claimed that difficulties are related to learning the multiple definitions of an angle, describing angles, measuring the size of angles, and conceiving different types of angles such as 0-line angles (an angle whose degree is 0 and 360 degrees), 1- line angles (an angle whose degree is 180 degrees), and 2-lines angles (an angle where both rays of the angle are visible) (Browning et al., 2008; Keiser, 2004; Mitchelmore & White, 1998).

While there are studies that shed light on students' difficulties with the concept of angles, there is limited research that explains how students learn the concept. Specifically, there is a lack of research that illuminates students' mental constructions of the concept of angles and how their mental constructions are related to their learning of more advanced concepts such as right triangles. In other words, there is need to expand the research in mathematics education concerning PSMTs' mental constructions of the concept of angles since angles are the fundamental concept to learn more advanced concepts. Therefore, this study was designed to describe and analyze pre-service secondary mathematics teachers' (PSMTs) mental constructions of the concept of angles from the perspective of **A**ction-**P**rocess-**O**bject-**S**chema (APOS) learning theory (Arnon, Cottrill, Dubinsky, Oktac, Roa Fuentes, Trigueros, & Weller, 2014; Asiala, Brown, DeVries, Dubinsky, Mathews, & Thomas, 1996; Clark, Cordero, Cottrill, Czarnocha, DeVries, John, Tolias, & Vidakovic, 1997; Dubinsky, 1991, 2010; Dubinsky & McDonald, 2001). The APOS framework was used to describe PSMTs' non-observable mental constructions of the concept of angles.

Study of the proposed research questions expands the limited literature on the learning of the concept of angles through the description of PSMTs' mental constructions of angles. The study also describes what kind of mental constructions of angles is needed in the right triangle context. Particularly, the descriptions can help researchers better understand PSMTs' levels of mental constructions—in terms of their mental actions, processes, objects, and schemas—of the concept of angles, which is foundational in the development of research-based curricula for the teaching and learning of angles.

Research Literature on the Concept of Angles

Existing studies on students' understanding of the concept of angles have considered elementary students' understanding of angle concept (Browning et al., 2008; Clements & Battista, 1989, 1990; Keiser, 2004; Mitchelmore & White, 1998). Although angle is a key concept within geometry, and learning the concept is a significant step to success in the discipline, all these studies indicated the limitations of those students' knowledge of angles. They claimed that students' difficulties in learning of the concept of angles are based on learning of the multiple definitions of an angle, describing angles, measuring the size of angles, and conceiving different types of angles such as 0-line angle, 1- line angle, and 2-line angle.

Many researchers proposed that three common representations are used to define an angle in mathematics education: an amount of turning between two lines (rotation), a pair of rays with a common point (vertex), and the region formed by the intersection of two lines (wedge) (Browning et al., 2008; Keiser, 2004; Mitchelmore & White, 2000). Particularly, Keiser (2004) compared sixth-grade students' definitions of angles to historical definitions of an angle. According to Keiser (2004), the multiple definitions of an angle creates confusion for students as they try to learn the basic concepts of angles; she stated, "all definition put limitations on the concept by focusing more heavily on one facet more than any of the others" (p. 289). Keiser (2004) also found that those students thought of angles as a vertex, rays, a corner, and a point, and they were confused when they tried to identify what part of angles exactly was being measured when they measured an angle.

Mitchelmore and White (2000) found that students in second to eighth grades struggled with identifying angles in physical situations. Their struggles stemmed from their need to identify both sides of angles. They claimed that the simplest angle concept was likely to be limited to situations where both sides of the angle were visible— 2-lines angles. However, when students were faced with a 1-line angle; they struggled to learn these situations as angles. Moreover, a 0-line angle is even more difficult for students to learn.

Clements and Battista (1989, 1990) proposed using a computer-based instructional method to teach the concept of angles. They specifically investigated the effects of computer programming in Logo to help third and fourth grades students develop and improve their learning. In Logo programming, students learn geometrical concepts by understanding and directing a turtle's movement, so based on the turtle's movement, Clements and Battista (1990) claimed that the program might be helpful "to elaborate on, and become cognizant of, the mathematics and problem-solving processes implicit in certain kinds of intuitive thinking" (p. 356), and to improve their understanding of the definition of angle. Browning et al. (2008) also moved beyond paper-and-pencil task in teaching the concept of angles, and they developed activities that include hands-on activities, graphing calculator applications, and computer software, Logo. Specifically, the researchers examined how technology-based activities helped sixth grade students to develop their knowledge of multiple representations of the concept of angles.

In another study, Moore (2013, 2014) investigated pre-calculus students' learning of angle measurement and trigonometry, and identified that quantitative and covariational reasoning are key factors to learn angle measurement and trigonometry in both unit circle and right triangle contexts. For Moore (2013), quantitative reasoning is involved in learning angle measurement. He proposed that an arc approach to angle measure can foster coherent experiences for students, and to improve their thinking in both unit circle and right triangle contexts. According to Moore (2013), students can be taught to connect angle measure to measuring arcs and conceive of the radius as a unit of measure. He concluded students needed to construct strong concepts of angles and angle measurement to conceptualize advanced concept such as unit circle and right triangle.

All these existing studies have revealed students' limited understanding of the concept of angles related to the multifaceted nature of the concept. In order to overcome students' difficulties of the concept, the researchers suggested that students should be taught using multiple definitions of an angle so that they will acquire and develop more comprehensive knowledge of angles. It is also more productive to present an angle by integrating multiple representations into instructional activities rather than simply giving a static definition of an angle (Keiser, 2004; Mitchelmore & White, 2000). In addition, Clements and Battista (1989, 1990) and Browning et al.'s (2008) studies demonstrated that the well-designed technology activities greatly facilitate students' development and exploration of angles and angle measurement. All these previous studies illustrated a need to gain better insight into adult learners'—PSMTs'—learning of the concept of angles as well as the relationships between these learners' levels of mental constructions of angles and more advanced concepts such as right triangles.

Theoretical Perspective

The APOS learning theory was used as a theoretical lens to determine PSMTs' mental constructions of the concept of angles. Dubinsky and his colleagues (Arnon et al., 2014; Asiala et al., 1996; Clark et al., 1997; Dubinsky, 1991; Dubinsky & McDonald, 2001) extended Piaget's theory of reflective abstraction, and applied it to advanced mathematical thinking to develop APOS learning theory. Their main goal in developing APOS theory was to create a model to investigate, analyze, and describe the level of students' mental constructions of a mathematical concept (Asiala et al., 1996). Specifically, a model is a description of how a schema for a specific mathematical concept develops and how the mental constructions of actions, processes, and objects can be used to construct the schema, and it is a useful guide for researchers to follow when investigating the levels of students' learning of a concept (Asiala et al., 1996). According to Dubinsky (1991), learning takes place in a student's mind through the construction of certain cognitive mechanisms, which includes mental constructions of actions, processes, objects and organizing them into schemas (See Figure 1). According to Asiala et al. (1996):

> An individual's mathematical knowledge is her or his tendency to respond to perceived mathematical situations by reflecting on problems and their solutions in a social context and by constructing or reconstructing mathematical actions, processes and objects and organizing these in schemas to use in dealing with situations. (p. 7)

Specifically, students use their existing knowledge of a physical or mental object to attempt to learn a new action. In order to learn a new concept, students carry out transformations by reacting to external cues that give exact details of which steps to take to perform an operation.

Then, an action might be interiorized into a process when an action is repeated, reflected upon, and/or combined with other actions. At the process level, students perform the same sort of transformations that they did at the action level, but the process level is not triggered by an external stimuli; the process level is an internal construction. Once students are able to reflect upon actions in a way that allows them to think about the process as an entity, they realize that transformations can be acted upon, and they are able to construct such transformations. In this case, the process is encapsulated into a cognitive object (Asiala et al., 1996). Students then organize the actions, processes, and objects, as well as prior schemas, into a new schema that accurately accommodates the new knowledge discovered from the mathematical problem.

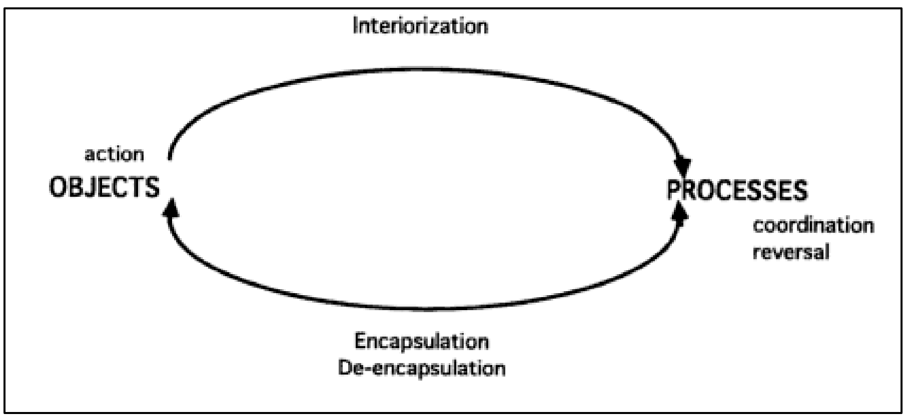


Figure 1. Schemas and their constructions (Adapted from Asiala et al., 1996)

Methodology

Because it was difficult to identify and describe PSMTs' non-observable mental constructions due to their highly internalized nature, this study utilized a series of controlled interviews, using the clinical interview methodology (Clements, 2000; Ericsson & Simon, 1993; Goldin, 2000; Newell & Simon, 1972) that is derived from Piaget's (1975) work. The main purpose of using clinical interviews in this study was to gather evidence of PSMTs' ways of reasoning and thinking and their level of mental constructions (Clements, 2000). Using the clinical interview methodology, I was able to use questioning to expose hidden structures and processes in their thoughts, ideas, and levels on the APOS theoretical framework as the interviews progress (Clements, 2000).

Participants and Settings

Participants of this study were PSMTs from two courses—The Teaching of Mathematics in Secondary Schools and Geometry—at a large public university in the Midwestern United States. One initial interview and five explanatory interviews were conducted with the participants. This paper explains a part of this large study and describes the initial interview, the first and second explanatory interviews, and a part of the third explanatory interview.

The initial interview session was used to select the required four to eight participants. The selection was based on the interested PSMTs' willingness to explain and articulate their thought processes, their experience with learning and teaching with technology, and their computer abilities since the tasks that were used in this study were adopted and developed in dynamic geometry software (DGS), GeoGebra. The initial interview was conducted with all volunteered seven participants in order to select required participants. Four—out of seven—PSMTs' (Linda, Kathy, Dana, and Jason) were selected to participate in the subsequent explanatory clinical interviews.

Both the initial interview and explanatory interviews were conducted in one-on-one sessions. One-on-one interviews were used because I anticipated that they would provide me with more reliable data than small group interviewing. Two different video cameras were used to record the interviews. One of the cameras was focused on the PSMTs and the researcher to capture the interactions; the other camera was zoomed in on the computer screen to record the

PSMTs' responses more closely. The recordings captured the PSMTs' mathematical utterances, gestures, and characteristics of speech.

Data Collection

The data collection included two separate parts: initial interview and five explanatory interviews. In initial interview, each of volunteered PSMTs was interviewed for half an hour, and each of them was given the same interview questions and tasks. Four PSMTs were selected based on the interests and their willingness to explain and articulate their thought processes, their experience with learning and teaching with technology, and their abilities of using GeoGebra. The initial interviews were not used to give insight into PSMTs' existing mental constructions.

After four PSMTs were selected, the explanatory interview sessions began. I conducted 60-minute, one-on-one interviews with each PSMT. The goal of the explanatory interviews was to help me gather evidence of PSMTs' ways of reasoning, thinking, and current knowledge of an angle, angle measurement, right triangles, relationship between angles and side lengths in a right triangle (RASR), and trigonometric ratios. The PSMTs' actions in response to the tasks, and articulation of their thought process and reasoning were used as evidence to investigate the PSMTs' mental actions, processes, objects, and schemas for the specific mathematical concept in each task.

The main goal of the first and the second explanatory interviews was to gain evidence of PSMTs' existing levels of mental constructions of the concept of angles and angle measurement. In order to explore evidence of PSMTs' existing level of the mental constructions of angles, PSMTs were given tasks that were adopted from Clements and Battista (1989, 1990) and Moore's (2010) studies and I built the tasks in GeoGebra. The goals of the third interview were to explore how PSMTs are connecting their mental constructions for the concepts of angles and angle measurement to construct knowledge of right triangles and to gain evidence of PSMTs' current levels of mental constructions in regards to RASR. The goal of the fourth and fifth interviews was to gain evidence of how PSMTs might reflect their knowledge of RASR to response the more advanced tasks, such as trigonometric ratios. This paper only focuses on the findings from the first and second interviews, and a part of the third interview.

Data Analysis

All of the data collected during the clinical interviews were analyzed using the APOS framework. This framework utilizes scripting, building a table that describes evidence points, transcribing the videos of the interview sessions, coding, describing PSMTs' levels of mental constructions of the concept of angles (Arnon et al., 2014; Asiala et al., 1996).

Once an explanatory interview session was completed, the video-recorded interviews were carefully transcribed; this was the preliminary level of the analysis. Once the transcription was completed, the video records were synced. The synced videotape data was vital for capturing moments of the PSMTs' verbal and nonverbal behaviors, speech characteristics, mathematical utterances, gestures, and sketches that they drew on GeoGebra. After compiling the interpretive notes of the synced video records, the transcript was scripted to find evidence of PSMTs' mental actions, processes, objects, or schemas for a particular concept. In this process, the researcher used a four-column table where the first column lists the code I assigned to an observed piece of evidence (as an action, a process, an object, or a schema), the second column contained my descriptions and reasons for my interpretations, the third column contained the original transcript of the event that leaded to my inferences, and the fourth column contained any extra notes. The combination, interactions, and coordination of the PSMTs' mental actions, processes, objects and schemas regarding the concept of angles and angle measurement were investigated and interpreted.

Results

The first and second explanatory interviews specifically were designed to investigate PSMTs' mental constructions of the concept of angles and angle measurement, specifically 0-line, 1-line and 2-line angles. In addition, in the third interview, one of the goals was to explain what kind of mental constructions of the concept of angles is needed in the context of right triangle. Following section describes PSMTs' mental constructions of the concept of angles and how their mental constructions of angles are related to their mental constructions of right triangles.

PSMTs' APOS Levels on Angles and Angle Measurement

PSMTs' mental constructions of 2-line angles. The first interview was started with PSMTs' drawings and definitions of an angle, and each PSMT defined an angle differently, but three representations were used to define an angle: angle as rotation, vertex, and wedge (See Table 1).

Table 1

PSMTs' definition of an angle

Name	Angle Definition	Representations of an angle
Linda	"An angle is a distance between two intersecting rays, so this could be viewed as line segments that would continue past the points [she was drawing and pointing out the arrows] (See Figure 2)."	Interior region between the intersection of two lines
Kathy	"My own words, OK. The definition of an angle is the… [long pause] It is the relationship between some line connected to the base, and the base itself [she was drawing two lines to define the angle] (See Figure 4.1). That is… It's like a distance, but not a distance of a straight line distance. It doesn't imply… This is like further versus farther. You know what I mean. With the u versus a. It is kind of the spread, I guess. A spread between two lines is an angle."	Rotation
Jason	"I defined it earlier, as the measure between two lines…um…but, I guess, it could… it's the position, I guess, that the lines are drawn from a single point. Not necessarily…well, I	Wedge and Interior region between the intersection of two lines

	guess that's still kind of the measurement. I don't know really how to explain it then…"	
Dana	"An angle in my mind would be a line or a vector in two different directions [indicating the different directions with the arrow] (See Figure 4.1)."	Interior region between the intersection

All PSMTs' definitions of an angle were directly related to their mental constructions of 2-line angles (See Table 1). All PSMTs drew 2-line angles to illustrate and discuss their definitions of an angle (See Figure 2). Throughout the interviews, it was determined that when PSMTs saw two segments or rays in a given figure, they could easily identify where the angle was, as well as they measure the angle. Therefore, it was inferred that they needed to see two segments or rays with one common point as an object to identify and measure that angle. The PSMTs' use of a physical object to act upon revealed the evidence of their action level for 2-line angles and angle measurement concepts.

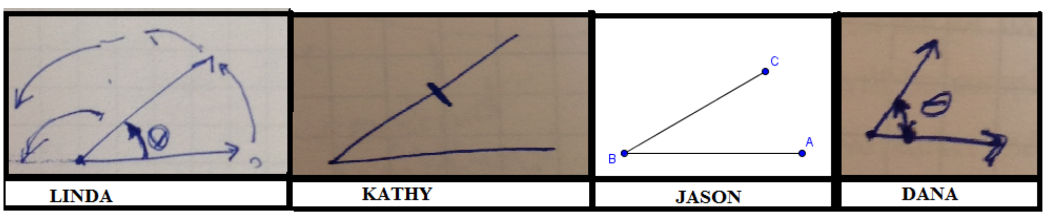


Figure 2. Linda, Kathy, Jason, and Dana's drawings to define an angle

To investigate whether they had reached the process level for 2-line angles, PSMTs were asked to draw an angle whose measure was greater than the angle that they previously drew. The PSMTs successfully drew a greater angle, and explained why the angle measure was greater. They generalized actions by explaining why the second angle they drew was greater than the first angle measure. In addition, when they were given a series of different figures (See Figure 3), they correctly identified all 2-line angles such as angle B, angle C, angle M. They also classified them as less than or greater than 90 degrees, 180 degrees, or 360 degrees. Their responses, namely generalizing their mental constructions and applying them to every object, revealed evidence of PSMTs' process level for 2-line angles and angle measurement.

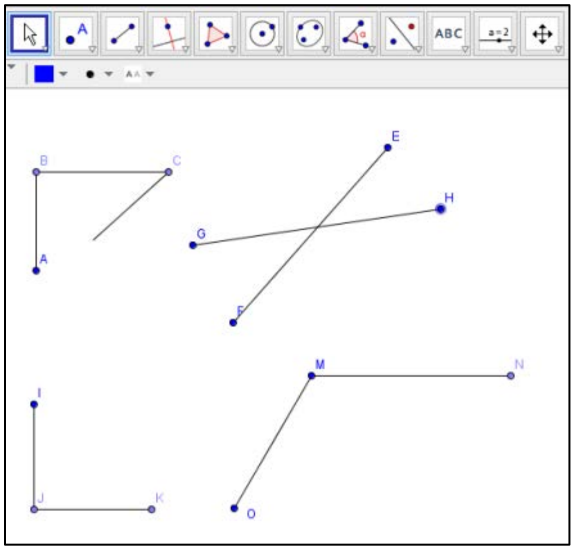


Figure 3. The task that includes 2-line angles

To investigate whether PSMTs had reached the object level, they were asked to compare the angles in a pair and explain how one angle could be described as a transformation of another angle (See Figure 4) (The task was adapted from Clements and Battista (1989)). The PSMTs acted on the figures using their mental constructions regarding 2-line angles. All four PSMTs proposed that the position of the angle could be transformed by moving the second angle on top of the first angle, and checking which had a larger measure. They acted on angles they identified and explained how one angle might be described as a transformation of another angle. Since object level is characterized by acting on a dynamic figure and realizing that transformations can be acted upon (Arnon et al., 2014; Asiala et al., 1996), the PSMTs' approaches were evidence that they were operating at the object level regarding 2-line angles and angle measurement.

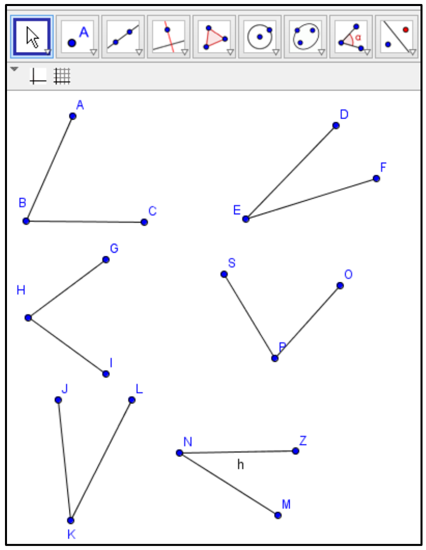


Figure 4. The task to investigate PSMTs' object level regarding 2-line angles

To explore the relationships between this general view of 2-line angles as related objects, PSMTs were asked to describe angles that measure between 1 degree and 34 degrees, or 180 degrees, 360, or *n* degrees as well as describe the relationships between these angle measurements (The task was adapted from Moore (2010)). All the PSMTs described the relationships between 1 degree angle and any other angle by describing them in terms of 1 degree angle. Jason's description was, "34 degrees is the one degree, 34 times. So, within the 34 degree angle, there's 34 one degree measures", and other participants' descriptions were similar to Jason's description. Particularly, all PSMTs described any angle's measurement as a transformation of another angle when they used two lines to draw. Since the object level is characterized by seeing the transformations can be acted upon it (Arnon et al., 2014; Asiala et al., 1996), PSMTs showed evidence of the object level of angle measurement for 2-line angles.

Evidence of schema for 2-line angles and angle measurement involves the use of action, process, and object levels in non-standard problem situation. All the PSMTs used their schemas and unpacked them, and reversed to the action, process, and objects levels as needed to solve non-routine tasks in the subsequent interviews. For instance, when they needed to use their 2-line angles schema to solve the task regarding right triangle context as it is mentioned in the following sections, they unpacked their schema to the action, process, or object levels to operate on the tasks. It shows that all the PSMTs demonstrated the evidence of their constructed schema associated with 2-lines angles and angle measurement.

PSMTs' mental constructions of 1-line angles. Mitchelmore and White (2000) and Keiser (2004) claimed that when students are faced with 1-line angles, they struggle to identify them as angles. They specifically look for a vertex point where the two lines connect and, not finding a vertex, conclude there is no angle. In order to explore PSMTs mental constructions of 1-line angles, they were asked to find the angles in given figures (See Figure 5). All PSMTs' responses were similar to those given by the students in Mitchelmore and White (2000) and Keiser's (2004) studies.

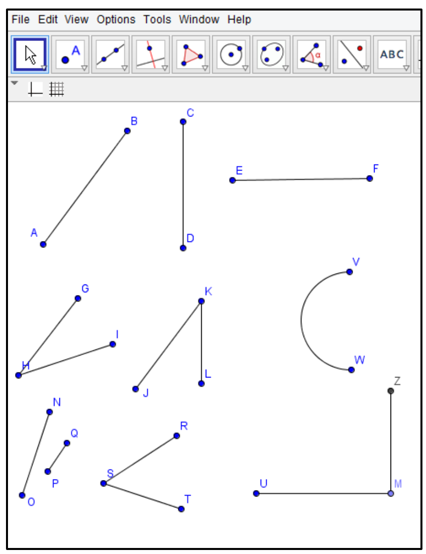


Figure 5. The task to investigate PSMTs' mental constructions for 1-line angle

When Linda was asked to find the angles in given figures (See Figure 5), she immediately identified all the 2-line angles in the figures. Then, she was asked whether there was an angle in any of the other figures. Pointing to the line segments such as AB, Linda proposed:

L: I am gonna say that these are simply line segments because [of] the way they were drawn, there are not multiple pieces intersecting.

Linda's reasoning was consistent with her definition of an angle as "a distance between two intersecting rays or line segments." She reasoned that there were no angles in line segments since "the line segment stopped at two points." Linda indicated that she needed to see two intersecting pieces—lines, rays, or line segments—to label the object as an angle. In other words, she needed to act on a physical object such as 2-line angles that provided specific details to determine whether there is an angle. Since the action level is characterized by using a physical object to act upon it, her approach showed evidence of the action level in terms of 1-line angles.

Kathy and Dana asserted that they needed to see a vertex point or imagine a vertex point to classify the line segments as angles. When Kathy was asked to find the angles in given figures, she immediately said that the line segments "were flat angles", and identified an imagined vertex point to define an angle. Dana also stated that line segments were just straight lines if you did not define the vertex point. Kathy and Dana's approaches to 1-line angles demonstrated evidence of their action levels for 1-line angles.

Jason's response was similar to other PSMTs' responses, but additionally he asserted that "a straight line includes an angle whose measure is 180 degrees." Jason initially proposed that angles could be defined in a straight line or in a line segment. He suggested, "I mean, I can see two angles because of one side and the other one, but ideally it's only one." He was relying on the fact that "a line segment includes a 180^{o} angle" and did not specifically show where the angle was even after prompting. His idea was a response to the presented physical objects in the figure based on the researcher's prompting like other participants. His response is also evidence that Jason was at the action level in terms of 1-line angles.

To elicit evidence of the process level for 1-line angles, the PSMTs were given a set of figures in Figure 6. They were asked to find and measure any angles that they could be determined. Linda and Dana's responses were similar while Kathy and Jason reasoned a bit differently. Both Linda and Dana indicated that they needed to see a vertex point or two intersecting line segments or rays to define a 1-line angle in a given figure. Both of them stated that they only determined and measured the angles on points A, B, I, and S (See Figure 6). Otherwise, there were no angles since they did not see a vertex point to define an angle. Both Linda and Dana's generalizing the actions for every condition was evidence of their process levels for 1-line angles (Arnon et al., 2014).

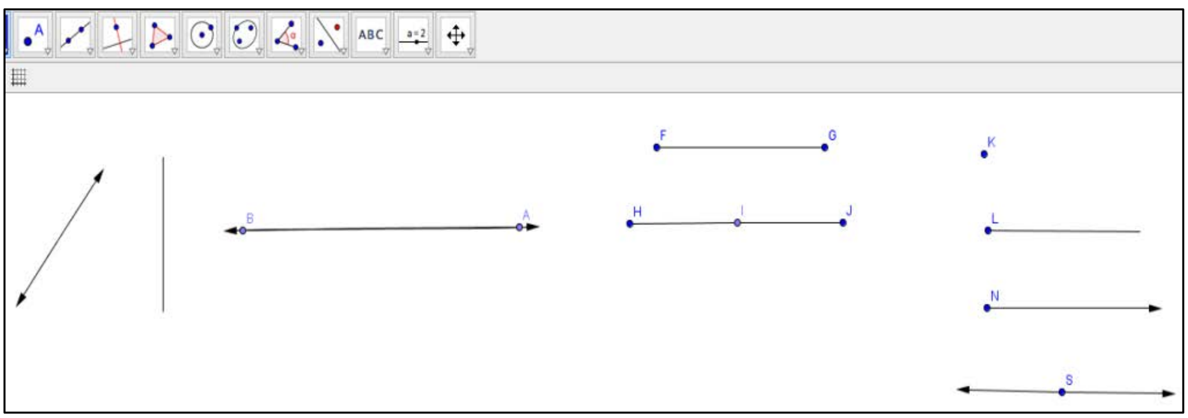


Figure 6. The task that includes 1-line angles

On the contrary of his response to the previous task, Jason then stated that an angle could be defined as the measurement between two lines, which meant that a line or line segment did not represent an angle for him. Similar to Linda and Dana's responses, Jason stated that there were angles on points A, B, I and S, because he could see the intersection of two lines at these points. As did the other PSMTs, he looked for a vertex point where the two lines connect to define a 1-line angle for every object, which provided evidence of the process level for 1-line angles and angle measurement.

Kathy immediately identified that a line or a line segment includes a 180^{o} angle while Linda and Dana indicated that there was no angle on a line or a line segment. Kathy imagined that there was a vertex point on line AB to identify the angle. However, she indicated that if she did not imagine a vertex, there would not be an angle. Kathy's generalization was evidence of her process level for 1-line angles and angle measurement.

To sum up, all of the PSMTs demonstrated evidence of the process level of 1-line angles to see an imagined or observable vertex point to posit the existence a measured for an angle. All required either an imagined or observable vertex point to posit the existence of measurement for an angle, which was evidence of the process level regarding 1-line angles and angle measurement. However, they did not provide any evidence that they reached the object or schema level. So, it was inferred that they remained at the process level in terms of 1-line angles and angle measurement.

PSMTs' mental constructions of 0-line Angles. 0-line angles are even more difficult to identify since no visible points or rays are given (Keiser, 2004; Mitchelmore & White, 2000). To investigate PSMTs' constructions of 0-line angles, they were asked to find angles (if there were any) in a set of figures which included a semi-circle, the letter B that was drawn using semi-circles and line segments, the letter S, and a circle (See Figure 7).

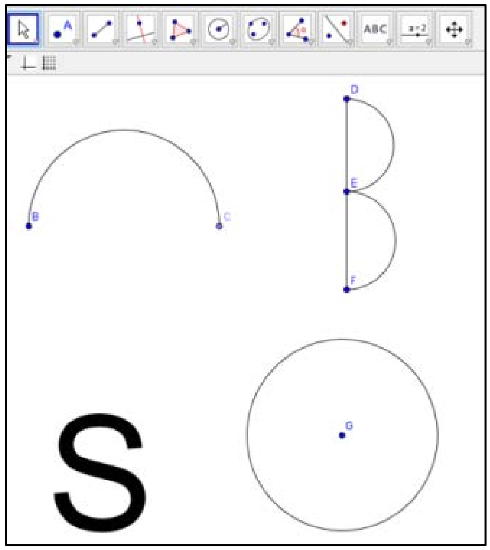


Figure 7. The task to investigate PSMTs' mental constructions for 0-line angle

Linda, Kathy, and Jason immediately asserted that there was no angle in the semi-circle. Linda went further and explained that "the semi-circle does not have an angle" *unless* they clarified a point represented at the center of the semi-circle and drew radius from to center to the points on the semi-circle (See Figure 8). Linda created a drawing to illustrate the possible angles by creating the center of the semi-circle and drawing the radius (See Figure 8). Linda, Kathy, and Jason also explained that there was no angle in the letter S because it is a curve.

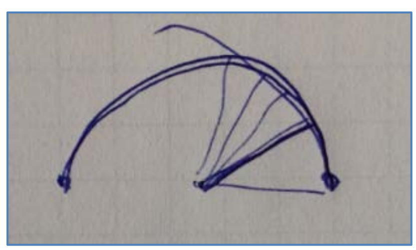


Figure 8. Linda's drawing to indicate whether there was an angle in semi-circle

Dana reasoned that both a semi-circle and the letter S include angles. She could identify semi-circles as angles without a given center because a semi-circle represents an arc for her. Dana initially indicated that there was no angle in S because there were not any points on S to define an angle. She also added that if she was allowed to define 2 points on S, she could identify an arc and an angle (See Figure 9). When Dana was asked to show those angles, she said there were many angles in S when she added the points on S as shown in Figure 9. Dana explained:

> D: Well, I mean as you draw, as you draw those [pointing out the arc lengths on S], then you would have some other points that you would define. So, I mean if you define some other points, then you would have some angle between those 2 points on that surface. And, again here, I determine the angle between those 2 points, and you could do the same thing on the outside.

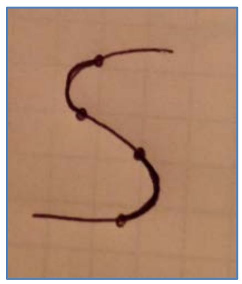


Figure 9. Redrawing of Dana's response to show the angles on S

In contrast to other PSMTs, a curve represented an angle for Dana if she was given or allowed to use two points on the curve.

After investigating whether a curve and a figure represented an angle for the PSMTs, they were asked whether a circle includes an angle. As in the previous task, Linda, Kathy, and Jason's responses were similar to each other. They all indicated that there was no angle in a circle since it is a curve and no two straight lines were visible. Kathy however suggested that if she added lines on the circle, she could determine the angle. Kathy stated:

> K: [pause] This is… It's all dependent on what you add, because, like, you can say that there are degrees here, you can, like, move fully a whole circle inside there, then there are 360 degrees, but it's… It's… they're no lines with which build an angle in this position.

Linda, Kathy, and Jason used a physical or mental object to reason about 0-line angles, which was evidence of the action level for 0-line angles. Dana initially suggested that “a circle has a continuous angle, never starts and never finishes.” When she was asked to show the angle in a given figure, she indicated, as did the other participants, that she needed points to create line segments to define an angle, which was also evidence of her action level for 0-line angles.

The PSMTs were given a set of circles in Figure 10 to further investigate their levels of mental constructions regarding 0-line angles. All PSMTs indicated that there was no angle in the first and the second circles since there were no lines presented. For the circle G, they proposed that radius GI was not enough to define an angle in a circle. Particularly, Linda suggested that if there were two line segments or rays in a circle where one was placed on the top of the other, it could be assumed that there was an angle in a given shape.

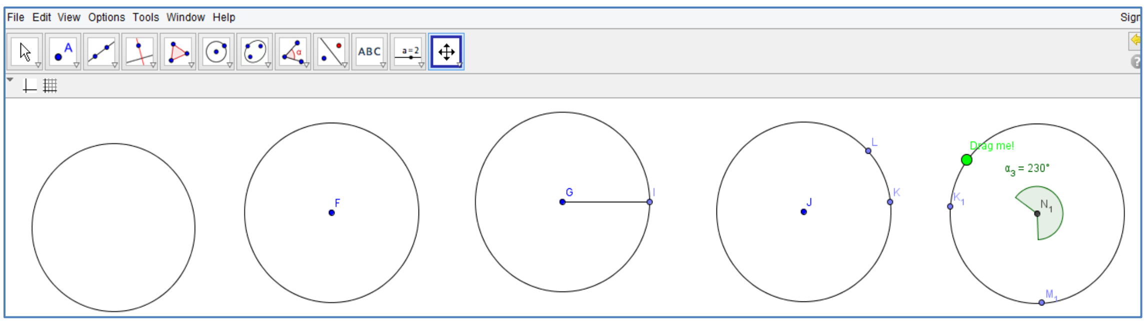


Figure 10. The task that includes 0-line angles

For the circle J, referring to their original definitions, all PSMTs suggested that if they were allowed to draw line segments JL and JK, they could define the angle. For the circle N_1, none determined an angle unless they were allowed to draw two line segments. Their responses are evidence that they always needed to draw or see two lines or line segments to interpret 0-line angles. This finding revealed that they internalized their actions and generalized their actions for every circle. In other words, they moved to having internal control over the objects. Their responses were evidence of the process level. However, their process level was limited because to identify the 0-line angles in a given figure, they needed a specific physical object that included two intersecting line segments in a circle. Otherwise, they proposed that there was no angle in the figure. Their responses did not provide any evidence that they reached the object or schema level. So, it was inferred that they remained at the process level in terms of 0-line angles and angle measurement.

Angles in Right Triangle Context

Throughout the third interview, I aimed to investigate what relationships PSMTs had between angles and side lengths in a right triangle. Additionally, one of the goals was to investigate how PSMTs' levels of mental constructions of angles were related to their mental constructions of right triangles.

I anticipated that PSMTs' mental constructions regarding right triangles were a special case of their mental constructions of any triangle since right triangles are a subset of all triangles. Therefore, to investigate how PSMTs' mental constructions of angles were related to their knowledge of right triangles, I began by asking them to explain what relationships they knew about angles and side lengths in a triangle. All PSMTs' responses were similar to each other, in that they immediately drew or imagined a triangle or right triangle to explain the relationships. For instance, when Dana was asked to explain relationships between angles and side lengths in any triangle, she immediately drew a right triangle and pointed out the Pythagorean Theorem (See Figure 11). She explained:

> D: Well, in a specific right triangle, then, if this were x and this is y, and this is z [referring to the right triangle she drew]. Then, not angles, but we know that x squared plus y squared is equal to z squared.

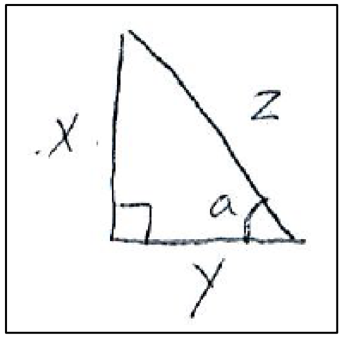


Figure 11. Dana's drawing of a right triangle

This approach to the task, namely physically drawing or imagining a triangle (or a right triangle) involving three angles comprised of rays to operate on and look for the relationships, was one the PSMTs used consistently in tasks throughout the interviews. Particularly, to draw or imagine the triangles—or right triangles, PSMTs recalled and applied their 2-line angle and angle measurement schema. For instance, Linda explained:

> L: Ok, so you have your triangle, you… It is composed of three angles, so each side on the triangle is actually a ray to two of the angles [meaning that two angles shared a common side in a triangle]. And so, the side length would be determined by how the angles are put together.

Linda, particularly, applied her mental constructions regarding 2-line angles schema to draw a triangle. Even though Linda as well as other PSMTs began to explain the relationships as a general case, they applied the same reasoning to right triangles later in the interview.

To further investigate how the PSMTs connected their mental constructions of angles, they were given a 30^0-60^0-90^0 triangle and asked to increase the 30^o angle to 35^o and identify the corresponding changes using paper and pencil (See Figure 12). All PSMTs increased the angle by acting on a physical object (on a right triangle) or imagining a right triangle even though they explained the changes differently .

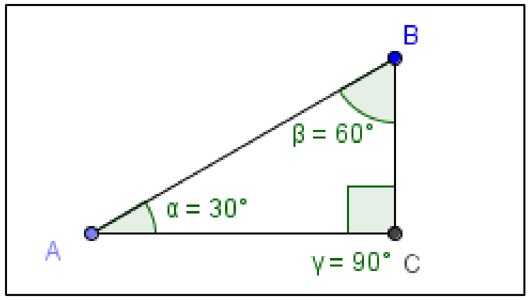


Figure 12. The task to investigate the role of angles in a right triangle context

For instance, Linda moved the point *A* horizontally to increase the angle to 35^0, and she preserved the right triangle and interpreted that other base angle decreased to 55^0. Additionally, Dana increased the angle to 35^0 by moving counterclockwise (See Figure 13). She first drew the given 30^0-60^0-90^0 right triangle using paper and pencil, and then she acted step by step on that triangle. She first preserved the right triangle and side length *AC* indicating “fixing this side” and increasing the angle *CAB* to 35^0 degrees by moving the side lengths *AB* towards counterclockwise. Then, Dana increased the side length *BC* (See Figure 13).

On the other hand, presumably, Kathy identified the changes in the right triangle before drawing the transformed triangle. When she was asked to explain what she thought about the task, she drew a right triangle and started to act on the physical object (See Figure 13). Kathy, specifically, increased angle *A* to 35^0 towards counterclockwise, preserving the right triangle by decreasing angle *B* to55^0. She further explained that she increased the angle counterclockwise as this rotation made the angle larger. She unpacked her 2-line angle measurement schema that she revealed before by indicating “the rotation makes the angle larger”. Similar to Kathy, Jason initially used an imagined triangle to increase the angle to 35^0, and then he drew the triangle (See Figure 13). He indicated that he increased the angle to 35^0 towards counterclockwise, and he decreased the other base angle to 55^0.

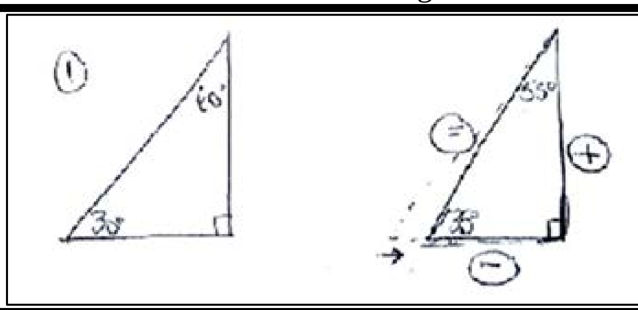
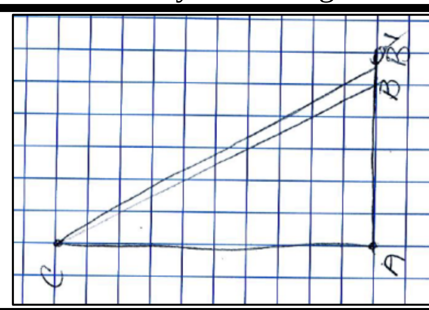
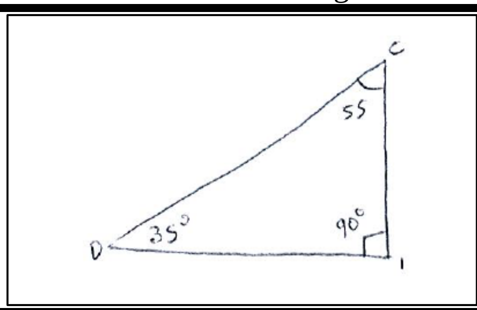
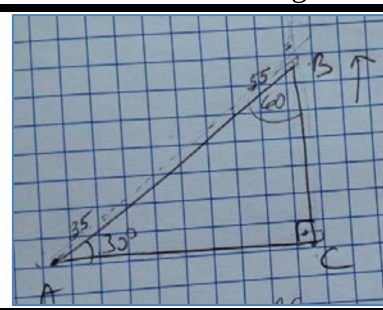
Linda's Drawing	Kathy's Drawing
	
Jason's Drawing	Dana's Drawing
	

Figure 13. PSMTs' drawings after they increased the 30^o angle to 35^o angle

The PSMTs' responses to the tasks reveal that their 2-line angles and angle measurement schema was enough to operate with right triangles. Like the participants in Keiser's (2004) and Mitchelmore and White's (1998) studies, the PSMTs did not struggle to act on angles where both sides were visible. In other words, although they remained at the process level and did not have full schema for 0-line and 1-line angles and angle measurement, their schema of 2-line angles and angle measurement was sufficient to reason about the tasks in right triangle context.

Discussion and Conclusion

Consistent with existing studies, all four PSMTs had limited knowledge of the concept of angles and angle measurement even though they were adult learners. Similar to the students in Mitchelmore and White (2000) and Keiser's (2004) studies, all PSMTs had a schema for 2-line angles and angle measurement. PSMTs were also less flexible on constructions of 1-line and 0-line angles and angle measurement as it applied to these angles. I inferred that their struggles with 1-line and 0-line angles stemmed from their descriptions of an angle. All the PSMTs defined and described an angle using two lines intersecting in a point, and all PSMTs indicated that they could easily determine angles where two rays were visible. However, when they were asked to find an angle in a given line segment or circle, they did not imagine two rays, and responded that there was no angle. Although PSMTs do not have a full schema regarding 0-line and 1-line angles and angle measurement, it was found that their constructions of 1-line and 0-line angles and angle measurement were not required in right triangles, and the schema level for 2-line angles was sufficient for constructions of right triangle context. In other words, object and schema level for 1-line and 0-line angles were not necessary to reason about right triangles since vertices or segment either were given or imagined in right triangles.

As Clements and Battista (1989, 1990) and Browning et al.'s (2008) suggested well-designed technology activities might enrich students' thinking and exploration of 0-line and 1-line angles. In order to help students reach higher levels of mental constructions regarding 1-line and 0-line angles and angle measurement, this study suggests that posing non-routine tasks about 0-line and 1-line angles and angle measurement in GeoGebra, would provide new opportunities to engage with different mathematical skills and levels of mental constructions. Particularly, dragging would be helpful for students to transform their mental constructions and determine the effects, differences, and properties of objects, and reach the schema level of 0-line and 1-line angles and angle measurement. Using the dragging aspect of GeoGebra and observing the relationships between 0-line, 1-line, and 2-line angles would have been helpful for students to reach higher levels of mental constructions regarding the concept of angles and angle measurement that can be applied to many different situations. Of course, the present study represents a step in this direction; it is essential to conduct further research to explore the roles of novel tasks in GeoGebra in construction schema for 0-line and 1-line angles as well as more advanced concepts.

In addition, Moore (2014) argued that that an arc approach to angle measure can foster coherent experiences for students in both unit circle and right triangle contexts. He suggested that to improve thinking in unit circle and right triangle contexts, students should be taught to relate angle arc measures and to consider the radius as a unit of measure of an arc. Moore (2014) further stated, “Developing meaning for angle measure and trigonometric functions that entail measuring arcs and lengths in a specified unit can also form important ways of reasoning for right triangle context” (p. 110). However, in this study, I investigated the characterization of PSMTs’ mental constructions regarding angle and angle measure constructions. This approach is different from Moore’s conclusions since findings from this study illustrate that a student can reason in the context of a right triangle without demonstrating mental constructions for arcs and arc lengths. All the PSMTs, for example, revealed evidence of 2-line angles and angle measurement schemas while they remained at the process level on 0-line and 1-line angles and angle measurement contexts. However, they were able to reason about right triangle tasks using their 2-line angles schema. Although the PSMTs did not reveal any evidence that they developed or applied meaning for angle measure that entail measuring arc and arc lengths as Moore (2014) suggested, their mental constructions of 2-line angles were enough to reason about the tasks which were designed in right triangle context. In particular, the level of constructions of 0-line angles may be the link between my findings and Moore’s (2014) conclusion. For instance, a level of mental constructions of 0-line angles that might lead to measuring arc and arc lengths could support important ways of reasoning in both right triangle and unit circle contexts, which remains an open question for future studies. In a larger context, the study is well situated within the canon of literature that addresses the crucial role of mediation through technology in which representations are embedded and executable (Moreno-Armella & Sriraman, 2005).

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